On the distribution of the length of the second row of a Young diagram under Plancherel measure

Jinho Baik
Courant Institute of Mathematical Sciences, New York
baik@cims.nyu.edu
Percy Deift
Courant Institute of Mathematical Sciences, New York
and
Institute for Advanced Study, New Jersey
deift@cims.nyu.edu
Kurt Johansson
Royal Institute of Technology, Sweden
kurtj@math.kth.se

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Abstract

We investigate the probability distribution of the length of the second row of a Young diagram of size N equipped with Plancherel measure. We obtain an expression for the generating function of the distribution in terms of a derivative of an associated Fredholm determinant, which can then be used to show that as $N \to \infty$ the distribution converges to the Tracy-Widom distribution [TW] for the second largest eigenvalue of a random GUE matrix. This paper is a sequel to [BDJ], where we showed that as $N \to \infty$ the distribution of the length of the first row of a Young diagram, or equivalently, the length of the longest increasing subsequence of a random permutation, converges to the Tracy-Widom distribution [TW] for the largest eigenvalue of a random GUE matrix.

1 Introduction

Let Y_N denote the set of all Young diagrams of size N. As is well known, Y_N may be viewed, equivalently, as the set of all partitions of N. We write $\mu = (\mu_1, \mu_2, \cdots) \in Y_N$ and/or $\mu \vdash N$, depending on the context. General references for properties of the Young diagrams are, for example, [Sa] and Section 5.1.4 in [Kn]. For $\mu \in Y_N$, d_μ is defined to be the number of (standard) Young tableaux of shape μ. By

Plancherel measure on Y_N, we mean the probability measure defined by

$$\mathrm{Prob}(\mu) := \frac{d_\mu^2}{N!}. \tag{1}$$

Under the Robinson-Schensted correspondence between S_N, the group of permutation of $\{1, 2, \cdots, N\}$, and the set of pairs of the Young tableaux of the same shape, Plancherel measure (1) is just the push forward of the uniform probability distribution on S_N. Let $l_N^{(k)}(\mu)$ denote the number of boxes in the k^{th} row of $\mu \in Y_N$. Schensted [Sc] showed that the length of the longest increasing subsequence of $\pi \in S_N$ is $l_N^{(1)}(\mu)$ where μ is the diagram of either of the Young tableaux corresponding to π under the Robinson-Schensted correspondence. Furthermore, Greene [Gr] obtained a combinatorial interpretations of $l_N^{(k)}$ for general $k \geq 2$, namely $l_N^{(1)}(\mu)+l_N^{(2)}(\mu)+\cdots+l_N^{(k)}(\mu)$ is the length of the longest k-increasing subsequence of π. (Recall that a k-increasing subsequence is a union of k disjoint increasing subsequences in π.) Similarly the sum of the lengths of the first k columns gives the length of the longest k-decreasing subsequence.

For $N \geq 1, n \geq 0$, define

$$q_{n,N}^{(k)} := \mathrm{Prob}\big(l_N^{(k)} \leq n\big) = \frac{1}{N!} \sum_{\substack{\mu \vdash N \\ l_N^{(k)}(\mu) \leq n}} d_\mu^2. \tag{2}$$

Set $q_{n,0}^{(k)} := 1$, for $n \geq 0$. Define the exponential generating function (or Poissonization) of $q_{n,N}^{(k)}$ by

$$\phi_n^{(k)}(\lambda) := \sum_{N=0}^{\infty} \frac{e^{-\lambda}\lambda^N}{N!} q_{n,N}^{(k)}. \tag{3}$$

First, we summarize the results of [BDJ] using the above notation. For $0 < t \leq 1$, let $u(x;t)$ be the unique solution of the Painlevé II equation

$$u_{xx} = 2u^3 + xu, \tag{4}$$

with the boundary condition

$$u(x;t) \sim -\sqrt{t}Ai(x) \quad \text{as} \quad x \to +\infty. \tag{5}$$

The proof of the existence and the uniqueness of this solution, as well as the asymptotics as $x \to -\infty$, can be found, for example, in [DZ2]. Define the Tracy-Widom [TW] distributions

$$F(x;t) = \exp\bigg(-\int_x^\infty (y-x)(u(y;t))^2 dy\bigg). \tag{6}$$

From the properties of $u(x;t)$ in [DZ2], it is easy to see that $F(x;t)$, $0 < t \leq 1$, is indeed a distribution function. One of the main results in [BDJ] is that for fixed $x \in \mathbb{R}$,

$$\lim_{N\to\infty} \mathrm{Prob}\bigg(\chi_N^{(1)} := \frac{l_N^{(1)} - 2\sqrt{N}}{N^{1/6}} \leq x\bigg) = F(x;1). \tag{7}$$

Convergence of the moments of $\chi_N^{(1)}$ is also proved in [BDJ], implying that

$$\lim_{N\to\infty}\frac{\mathrm{Var}\big(l_N^{(1)}\big)}{N^{1/3}}=\int_{-\infty}^{\infty}t^2dF(t;1)-\left(\int_{-\infty}^{\infty}tdF(t;1)\right)^2=0.8132\cdots. \tag{8}$$

and

$$\lim_{N\to\infty}\frac{\mathrm{Exp}\big(l_N^{(1)}\big)-2\sqrt{N}}{N^{1/6}}=\int_{-\infty}^{\infty}tdF(t;1)=-1.7711\cdots. \tag{9}$$

The above results are intimately connected to the Gaussian Unitary Ensemble (GUE) of random matrix theory. In GUE, one considers $N\times N$ Hermitian matrices, and the probability density for the eigenvalues in an infinitesimal multi-interval about points $x_1,\cdots,x_N$ is given by

$$Z_N^{-1}e^{-\sum_{j=1}^N x_j^2}\prod_{1\le j<k\le N}|x_j-x_k|^2dx_1\cdots dx_N, \tag{10}$$

where Z_N is the normalization constant. Let $\lambda_{1^{\mathrm{st}}}(M)$ be the largest eigenvalue of M in GUE. In 1994 in [TW], Tracy and Widom showed that if one scales $\lambda_{1^{\mathrm{st}}}^{\mathrm{sc}}:=\big(\lambda_{1^{\mathrm{st}}}-\sqrt{2N}\big)\sqrt{2}N^{1/6}$, then for any fixed $x\in\mathbb{R}$,

$$\lim_{N\to\infty}\mathrm{Prob}\big(\lambda_{1^{\mathrm{st}}}^{\mathrm{sc}}\le x\big)=F(x;1). \tag{11}$$

In other words, properly centered and scaled, the length of the longest increasing subsequence for $\pi\in S_N$ (or $l_N^{(1)}$ under Plancherel measure on Y_N), behaves statistically for large N like the largest eigenvalue of a random GUE matrix. Moreover, in [TW] the authors also computed the limiting distribution of the k^{th} largest eigenvalue $\lambda_{k^{\mathrm{th}}}$ of a random GUE matrix for general $k\ge 2$. Indeed, again scaling $\lambda_{k^{\mathrm{th}}}^{\mathrm{sc}}:=\big(\lambda_{k^{\mathrm{th}}}-\sqrt{2N}\big)\sqrt{2}N^{1/6}$, Tracy and Widom showed that for fixed $x\in\mathbb{R}$, and for $k\ge 2$,

$$\lim_{N\to\infty}\mathrm{Prob}\big(\lambda_{k^{\mathrm{th}}}^{\mathrm{sc}}\le x\big)=\lim_{N\to\infty}\mathrm{Prob}\big(\lambda_{(k-1)^{\mathrm{th}}}^{\mathrm{sc}}\le x\big)+\frac{1}{(k-1)!}\left(-\frac{\partial}{\partial t}\right)^{(k-1)}\bigg|_{t=1}F(x;t). \tag{12}$$

In this paper, we prove that $l_N^{(2)}$ behaves statistically, as $N\to\infty$, like the second largest eigenvalue of a random GUE matrix. This result was conjectured in [BDJ] : the obvious analogous result should be true for all the rows. For the second row, the conjecture was strongly supported by Monte Carlo simulations of Odlyzko and Rains. To compute the asymptotics of $l_N^{(2)}$, we first obtain an expression for the generating function $\phi_n^{(2)}(\lambda)$ in terms of a derivative of a Fredholm determinant. For the case $k=1$ in [BDJ], it was already known (see, for example, [Ge], [Ra]) that the generating function $\phi_n^{(1)}(\lambda)$ is a Toeplitz determinant :

$$\phi_n^{(1)}(\lambda)=e^{-\lambda}\det\big(T_{n-1}\big),\qquad n\ge 0, \tag{13}$$

where $T_{n-1}=\big((T_{n-1})_{j,k}\big)_{0\le j,k\le n-1}$ is the $n\times n$ Toeplitz matrix with respect to the weight $e^{2\sqrt{\lambda}\cos\theta}d\theta/(2\pi)$,

$$T_{n-1}=\left(\int_0^{2\pi}e^{-i(j-k)\theta}e^{2\sqrt{\lambda}\cos\theta}\frac{d\theta}{2\pi}\right)_{0\le j,k\le n-1},\qquad n\ge 1, \tag{14}$$

and T_{-1} is the 1×1 matrix with entry equal to 1. Therefore the main part of the calculation in [BDJ] was the computation of the asymptotics of $\phi_n^{(1)}(\lambda)$ as $\lambda, n \to \infty$. In this paper, however, we focus on deriving an appropriate expression for the generating function $\phi_n^{(2)}(\lambda)$. As we will explain, the asymptotics as $\lambda, n \to \infty$ can then be obtained in a similar manner to [BDJ]. An interesting by-product of the calculation of $\phi^{(2)}(\lambda)$ is a new expression for $\phi^{(1)}(\lambda)$ (see (17) below).

We state our results. Set

$$\varphi(z) := e^{\sqrt{\lambda}(z-z^{-1})}. \tag{15}$$

and let Σ be the unit circle in the complex plane oriented counterclockwise. Let K_n be the integral operator acting on $L^2(\Sigma, |dw|)$, whose kernel is defined by

$$\mathrm{K}_n(z,w) := \frac{z^{-n}w^n - \varphi(z)\varphi(w)^{-1}}{2\pi i(z-w)}, \qquad (\mathrm{K}_n\, f)(z) = \int_\Sigma \mathrm{K}_n(z,w) f(w) dw. \tag{16}$$

Theorem 1. *For $n \geq 0$, we have*

$$\phi_n^{(1)}(\lambda) = 2^{-n} \det(I - \mathrm{K}_n), \tag{17}$$

and

$$\phi_{n+1}^{(2)}(\lambda) = \phi_n^{(1)}(\lambda) + \left(-\frac{\partial}{\partial t}\right)\bigg|_{t=1} \left[(1+\sqrt{t})^{-n} \det(I - \sqrt{t}\,\mathrm{K}_n)\right]. \tag{18}$$

Theorem 2. *For fixed $x \in \mathbb{R}$, we have*

$$\lim_{N\to\infty} \mathrm{Prob}\left(\chi_N^{(2)} := \frac{l_N^{(2)} - 2\sqrt{N}}{N^{1/6}} \leq x\right) = F^{(2)}(x), \tag{19}$$

where $F^{(2)}(x) = F(x;1) + \left(-\frac{\partial}{\partial t}\right)\big|_{t=1} F(x;t)$ is the Tracy-Widom second eigenvalue distribution formula given in (12) above.

As in [BDJ], we also have convergence of the moments. Indeed, let $\chi^{(2)}$ be a random variable with distribution function $F^{(2)}$. The we have the following result.

Theorem 3. *For $m = 1, 2, \cdots,$*

$$\lim_{N\to\infty} \mathrm{Exp}\big((\chi_N^{(2)})^m\big) = \mathrm{Exp}\big((\chi^{(2)})^m\big). \tag{20}$$

These results show that $l_N^{(2)}$ behaves statistically for large N like the second largest eigenvalue of a random GUE matrix, under appropriate centering and scaling. Furthermore, in view of the previous remarks, by the Robinson-Schensted correspondence these results show that, after appropriate centering and scaling, the difference between the length of the longest 2-increasing subsequence and the length of the longest increasing subsequence of a random permutation of N numbers, also behaves statistically for large N like the second largest eigenvalue of a random GUE matrix.

As indicated above, the bulk of the paper is devoted to proving formula (18) for $\phi_{n+1}^{(2)}(\lambda)$. We indicate briefly in Section 6 below how Theorems 2 and 3 then follow by the Riemann-Hilbert/steepest descent methods of [BDJ] : further details on the computations will appear in a later publication. In [BDJ], the authors express $\phi_n^{(1)}(\lambda)$ via (13) in terms of the solution of a Riemann-Hilbert problem (RHP) for polynomials orthogonal with respect to the weight $e^{2\sqrt{\lambda}\cos\theta}d\theta/(2\pi)$ on the unit circle (this RHP is the analog for orthogonal polynomials on the circle of the RHP introduced in [FIK] for orthogonal polynomials on the line), and then apply the steepest descent method for RHP's introduced by Deift and Zhou in [DZ1], further developed in [DZ2], [DVZ1], and finally placed in a systematic form by Deift, Venakides and Zhou in [DVZ2], to compute the asymptotics as $\lambda, n \to \infty$. A general reference for RHP's is, for example, [CG]. The calculations in [BDJ] have many similarities to the calculations in [DKMVZ]. The methods of [BDJ] apply here because the operator K_n in (16) is an example of a so-called integrable operator, whose resolvent can be computed in terms of a canonically associated RHP (see Section 2 below). Integrable operators were introduced as a distinguished class by Its, Izergin, Korepin and Slavnov in [IIKS], and have since been applied to a broad and rapidly growing array of problems in pure and applied mathematics (see, for example, [De]).

The proof of Theorem 1 is based, in large part, on manipulations of RHP's. The proof of equality (17) for $\phi_n^{(1)}(\lambda)$ is given in Section 2 (see Proposition 6). In Section 2, the RHP's $(v_Y(\cdot\,;k),\Sigma)$ and $(v(\cdot\,;k;t),\Sigma)$ are also introduced and their connections to the Fredholm determinant $\det(I-\mathrm{K}_N)$ are established. In Section 3, to prove (18) for $\phi_{n+1}^{(1)}(\lambda)$, we first use the Frobenius-Young formula for d_μ in (1) to obtain an intermediate form for $\phi_{n+1}^{(2)}(\lambda)$ in terms of the inverse of a Toeplitz matrix together with certain binomial sums (see Proposition 7). The calculation of this intermediate form is similar to the derivation of (13) above in the Appendix in [BDJ]. The identification of this intermediate form with the right hand side of (18) is made first when $n=0$ (Section 4), and then for general $n \geq 1$ (Section 5). Finally, in Section 6, we indicate how to prove Theorems 2 and 3 following the methods in [BDJ]. In the Appendix, we discuss the spectral properties of the operator K_n in (16), and the (unique) solvability of the RHP (24) below.

The proof that we give for the basic formula (18) appears rather ad hoc. At the end of the paper, in Section 6, we provide a motivation for our calculations.

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2 Riemann-Hilbert problems (RHP's)

As above, let Σ denote the unit circle in the complex plane, oriented counterclockwise. Set

$$\psi(z) := e^{\sqrt{\lambda}(z+z^{-1})}. \tag{21}$$

In this paper, we use the following two (matrix) RHP's. First, for $k \geq 0$, let $Y(z;k)$, a 2×2 matrix-valued function of z, be the solution of the RHP $(v_Y(\cdot\,;k),\Sigma)$,

$$\begin{cases} Y(z;k) \quad \text{is analytic in} \quad z \in \mathbb{C}\setminus\Sigma, \\ Y_+(z;k) = Y_-(z;k)\begin{pmatrix} 1 & \frac{1}{z^k}\psi(z) \\ 0 & 1 \end{pmatrix}, \quad \text{on} \quad z\in\Sigma, \\ Y(z;k)\begin{pmatrix} z^{-k} & 0 \\ 0 & z^k \end{pmatrix} = I + O(\frac{1}{z}) \quad \text{as} \quad z\to\infty. \end{cases} \tag{22}$$

The notation $Y_+(z;k)$ (resp., Y_-) denotes the limiting value $\lim_{z'\to z} Y(z';k)$ with $|z'|<1$ (resp., $|z'|>1$). Note that k plays the role of an external parameter in (22) ; in particular, the term $O(\frac{1}{z})$ does not imply a uniform bound in k. For a general RHP, the existence and uniqueness of the solution is, of course, not clear a priori. But for the case at hand, we can simply write down the (unique) solution explicitly in terms of orthogonal polynomials, as follows.

Let $\pi_n(z) = z^n + \sum_{p=0}^{n-1}\eta_p^n z^p$ denote the n-th monic orthogonal polynomial with respect to the measure $\psi(z)dz/(2\pi i z)$ on the unit circle, and introduce the polynomial $\pi_n^*(z) := z^n\overline{\pi_n}(1/z) = z^n(z^{-n} + \sum_{p=0}^{n-1}\overline{\eta_p^n}z^{-p}$ (see [Sz]). For the measure at hand, $\psi(z)dz/(2\pi i z) = e^{2\sqrt{\lambda}\cos\theta}d\theta/(2\pi)$, all the coefficients of $\pi_n(z)$ are real and $\pi_n^*(z) = z^n\pi_n(1/z)$. The solution of the RHP $(v_Y(\cdot\,;k),\Sigma)$ is given by the following formula (see Lemma 4.1 in [BDJ] ; here we change the notation $f(z)$ to $\psi(z)$ and use the monic orthogonal polynomial $\pi_k(z)$ instead of the normalized orthogonal polynomials $p_k(z)$),

$$Y(z;k) = \begin{pmatrix} \pi_k(z) & \int_\Sigma \frac{\pi_k(s)}{s-z}\frac{\psi(s)ds}{2\pi i s^k} \\ -\kappa_{k-1}^2\pi_{k-1}^*(z) & -\kappa_{k-1}^2\int_\Sigma \frac{\pi_{k-1}^*(s)}{s-z}\frac{\psi(s)ds}{2\pi i s^k} \end{pmatrix}, \qquad k\geq 1. \tag{23}$$

For $k=0$, the solution is given by (82) below.

Let $\varphi(z)$ be defined as in (15). For $0<t\leq 1$ and for $k\geq 0$, the second RHP $(v(\cdot\,;k;t),\Sigma)$ is to find $m(z;k;t)$ satisfying

$$\begin{cases} m(z;k;t) \quad \text{is analytic in} \quad z\in\mathbb{C}\setminus\Sigma, \\ m_+(z;k;t) = m_-(z;k;t)\begin{pmatrix} 1-t & -\sqrt{t}z^{-k}\varphi(z)^{-1} \\ \sqrt{t}z^k\varphi(z) & 1 \end{pmatrix} \quad \text{on } z\in\Sigma, \\ m(z;k;t)\to I \quad \text{as} \quad z\to\infty. \end{cases} \tag{24}$$

In this RHP, there are two external parameters k and t. For $t=1$, this RHP is equivalent to the RHP $(v_Y(\cdot\,;k),\Sigma)$, in the sense that a solution of one RHP implies a solution of the other RHP. Indeed one

can easily check that

$$m(z;k;1) = \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Y(z;k) \begin{pmatrix} e^{\sqrt{\lambda}z} & 0 \\ 0 & e^{-\sqrt{\lambda}z} \end{pmatrix}, & |z|<1, \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Y(z;k) \begin{pmatrix} z^{-k}e^{\sqrt{\lambda}z^{-1}} & 0 \\ 0 & z^k e^{-\sqrt{\lambda}z^{-1}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & |z|>1, \end{cases} \tag{25}$$

solves (24) if and only if $Y(z;k)$ solves (22). The (unique) solvability of the RHP (24) for $0<t<1$ is discussed in the Appendix. As noted earlier, the motivation for the introduction of (24) for $t<1$ is given in Section 6 below.

The solution $m(z;k;t)$ of the RHP $(v(z;k;t),\Sigma)$ is related to the Fredholm determinant of the integral operator K_n in (16) acting on $L^2(\Sigma,|dw|)$. For each n, it is easy to verify that K_n is trace class and hence $\det(I-\mathrm{K}_n)$ exist. The proof of the following Lemma is modeled on Proposition 6.13 in [DIZ].

Lemma 4. *Let $m(z;k;t)$ be the solution of the RHP $(v(z;k;t),\Sigma)$ given in (24), and let K_k be defined as above. If we denote the 11-component of $m(z;k;t)$ by $m_{11}(z;k;t)$, we have at $z=0$,*

$$m_{11}(0;k;t) = (1+\sqrt{t})\frac{\det(I-\sqrt{t}\,\mathrm{K}_{k-1})}{\det(I-\sqrt{t}\,\mathrm{K}_k)}, \qquad k\geq 0. \tag{26}$$

Proof. The operator K_n has norm less than or equal to 1 and 1 is not an eigenvalue of K_n (see Appendix). So $\det(I-\sqrt{t}\,\mathrm{K}_k)$ never vanishes for any k, $0<t\leq 1$.

First note that

$$\mathrm{K}_{k-1}(z,w) = \mathrm{K}_k(z,w) + \frac{1}{2\pi i} z^{-k}w^{k-1} =: \mathrm{K}_k(z,w) + E_k(z,w). \tag{27}$$

Then we have

$$\frac{\det(I-\sqrt{t}\,\mathrm{K}_{k-1})}{\det(I-\sqrt{t}\,\mathrm{K}_k)} = \det\left(I - \frac{1}{I-\sqrt{t}\,\mathrm{K}_k}\sqrt{t}E_k\right). \tag{28}$$

Since E_k is a rank 1 operator,

$$\frac{\det(I-\sqrt{t}\,\mathrm{K}_{k-1})}{\det(I-\sqrt{t}\,\mathrm{K}_k)} = 1 - \mathrm{tr}\left(\frac{1}{I-\sqrt{t}\,\mathrm{K}_k}\sqrt{t}E_k\right) = 1 - \frac{\sqrt{t}}{2\pi i}\int_\Sigma \left(\frac{1}{I-\sqrt{t}\,\mathrm{K}_k} f_1\right)(z)\cdot z^{k-1}dz, \tag{29}$$

where $f_1(z)=z^{-k}$ as in (33) below.

On the other hand, we define $M(z,k;t)$ as follows,

$$M(z;k;t) := \begin{cases} m(z;k;t)\begin{pmatrix} 1/(1+\sqrt{t}) & 0 \\ 0 & 1+\sqrt{t} \end{pmatrix}, & |z|<1 \\ m(z;k;t), & |z|>1. \end{cases} \tag{30}$$

Then it is easy to check that $M(z;k;t)$ solves a new RHP $(V(\cdot\,;k;t),\Sigma)$,

$$\begin{cases} M(z;k;t) \quad \text{is analytic in} \quad \mathbb{C}\setminus\Sigma, \\ M_+(z;k;t) = M_-(z;k;t)\begin{pmatrix} 1-\sqrt{t} & -\sqrt{t}(1+\sqrt{t})z^{-k}\varphi(z)^{-1} \\ \frac{\sqrt{t}}{1+\sqrt{t}}z^k\varphi(z) & 1+\sqrt{t} \end{pmatrix} \quad \text{on } z\in\Sigma, \\ M(z;k;t)\to I \quad \text{as} \quad z\to\infty. \end{cases} \tag{31}$$

Note that the jump matrix $V(z;k;t)$ can be written in the form

$$V(z;k;t) = I - 2\pi i\sqrt{t}f(z;k;t)\big(g(z;k;t)\big)^T, \tag{32}$$

where f, g are column vectors defined by

$$f(z;k;t) = (f_1, f_2)^T = \big(z^{-k}, -\frac{1}{1+\sqrt{t}}\varphi(z)\big)^T, \quad g(z;k;t) = (g_1, g_2)^T = \frac{1}{2\pi i}\big(z^k, (1+\sqrt{t})\varphi(z)^{-1}\big)^T. \tag{33}$$

From the general theory of RHP's, the solution $M(z;k;t)$ of the RHP $(V(z;k;t), \Sigma)$ satisfies

$$\begin{aligned} M(z;k;t) &= I + \frac{1}{2\pi i}\int_\Sigma \frac{M_+(s;k;t)\big(I - V(s;k;t)^{-1}\big)}{s-z}ds \\ &= I - \frac{\sqrt{t}}{2\pi i}\int_\Sigma M_+(s;k;t)\begin{pmatrix} 1 & (1+\sqrt{t})s^{-k}\varphi(s)^{-1} \\ -\frac{1}{1+\sqrt{t}}s^k\varphi(s) & -1 \end{pmatrix}\frac{ds}{s-z}. \end{aligned} \tag{34}$$

Recalling the definition of $f(z;k;t)$, we have

$$M_{11}(0;k;t) = 1 - \frac{\sqrt{t}}{2\pi i}\int_\Sigma \big(M_+(z;k;t)f(z;k;t)\big)_1 \cdot z^{k-1}dz. \tag{35}$$

Now from the theory of integrable operators (see for example [De]), the integral operator S_k acting on $L^2(\Sigma, |dw|)$ with kernel

$$S_k(z,w) := \frac{(f(z;k;t))^T g(w;k;t)}{z-w}, \tag{36}$$

satisfies the relation

$$\left(\frac{1}{I - \sqrt{t}S_k}S_k\right)(z,w) = \frac{\big(F(z;k;t)\big)^T G(w;k;t)}{z-w}, \tag{37}$$

where F, G are column vectors given by

$$\begin{aligned} F(z;k;t) &= \left(\frac{1}{I-\sqrt{t}S_k}f\right)(z) = M_+(z;k;t)f(z;k;t), \\ G(w;k;t) &= \left(\frac{1}{I-\sqrt{t}S_k}g\right)(w) = \big(M_+(w;k;t)^T\big)^{-1}g(w;k;t). \end{aligned} \tag{38}$$

But from the definitions of f and g in (33),

$$S_k(z,w) = \mathrm{K}_k(z,w). \tag{39}$$

Therefore (29), (35) and the first part of (38) give us that

$$M_{11}(0;k;t) = \frac{\det(I - \sqrt{t}\,\mathrm{K}_{k-1})}{\det(I - \sqrt{t}\,\mathrm{K}_k)}. \tag{40}$$

The relation (30) completes the proof. □

Now for fixed t, λ, we compute the asymptotics of $\det(I - \sqrt{t}\,\mathrm{K}_p)$ as $p \to \infty$. The calculation below is similar to that of [De] where a new proof of Szegö's strong limit theorem is given.

Lemma 5. *For fixed $0 < t \le 1$ and $\lambda > 0$, we have*

$$\lim_{p\to\infty} (1+\sqrt{t})^{-p} \det(I - \sqrt{t}\,\mathrm{K}_p) = 1. \tag{41}$$

Proof. First note that

$$\log\det(I - \sqrt{t}\,\mathrm{K}_p) = -\int_0^{\sqrt{t}} \operatorname{tr}\left(\frac{1}{I - s\,\mathrm{K}_p}\,\mathrm{K}_p\right) ds. \tag{42}$$

From (37) and (39), the right hand side of (42) is given by

$$-\int_0^{\sqrt{t}} \operatorname{tr}\left(\frac{\big(F(z;p;s^2)\big)^T G(w;p;s^2)}{z-w}\right) ds = -\int_0^{\sqrt{t}} ds \int_{\Sigma} \big(F'(z;p;s^2)\big)^T G(z;p;s^2) dz \tag{43}$$

where the prime $'$ indicates differentiation with respect to z. From (38), in order to compute the asymptotics of $\det(I - \sqrt{t}\,\mathrm{K}_p)$, we need the asymptotics of $M_+(z;p;s^2)$ as $p \to \infty$ uniformly in $0 < s \le \sqrt{t}$.

Note that the jump matrix $V(z;p;s^2)$ has the following factorization

$$V(z;p;s^2) = \begin{pmatrix} 1 & -sz^{-p}\varphi^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{1+s} & 0 \\ 0 & 1+s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{s}{(1+s)^2} z^p \varphi & 1 \end{pmatrix}. \tag{44}$$

Let $0 < \rho < 1$ be any number. Define

$$\begin{cases} \tilde{M}(z) := M(z;p;s^2), & |z| < \rho, \\ \tilde{M}(z) := M(z;p;s^2) \begin{pmatrix} 1 & 0 \\ \frac{s}{(1+s)^2} z^p \varphi & 1 \end{pmatrix}^{-1}, & \rho < |z| < 1, \\ \tilde{M}(z) := M(z;p;s^2) \begin{pmatrix} 1 & -sz^{-p}\varphi^{-1} \\ 0 & 1 \end{pmatrix}, & 1 < |z| < \rho^{-1}, \\ \tilde{M}(z) := M(z;p;s^2), & |z| > \rho^{-1}, \end{cases} \tag{45}$$

and set

$$\begin{cases} \tilde{V}(z) = \begin{pmatrix} 1 & 0 \\ \frac{s}{(1+s)^2} z^p \varphi & 1 \end{pmatrix}, & |z| = \rho, \\ \tilde{V}(z) = \begin{pmatrix} \frac{1}{1+s} & 0 \\ 0 & 1+s \end{pmatrix}, & |z| = 1, \\ \tilde{V}(z) = \begin{pmatrix} 1 & -sz^{-p}\varphi^{-1} \\ 0 & 1 \end{pmatrix}, & |z| = \rho^{-1}. \end{cases} \tag{46}$$

Then $\tilde{M}$ solves the new RHP $(\tilde{V}, \tilde{\Sigma})$ where $\tilde{\Sigma} := \{|z| = \rho\} \cup \{|z| = 1\} \cup \{|z| = \rho^{-1}\}$, oriented counterclockwise on each of the three circles,

$$\begin{cases} \tilde{M}(z) \quad \text{is analytic in } \mathbb{C} \setminus \tilde{\Sigma}, \\ \tilde{M}_+(z) = \tilde{M}_-(z)\tilde{V}(z), \quad \text{on } z \in \tilde{\Sigma}, \\ \tilde{M}(z) \to I, \qquad \text{as } z \to \infty. \end{cases} \tag{47}$$

Here $\tilde{M}_+(z) = \lim_{z'\to z, |z'|<|z|} \tilde{M}(z')$ and $\tilde{M}_-(z) = \lim_{z'\to z, |z'|>|z|} \tilde{M}(z')$ for z on each of the three circles. Now as $p \to \infty$, we see that $\tilde{V}(z) \to I$ for $z \in \{|z| = \rho\} \cup \{|z| = \rho^{-1}\}$. Moreover the convergence is exponential at a rate which is uniform in z and s. Therefore it follows that $\tilde{M} \to \tilde{M}^\infty$ where $\tilde{M}^\infty$ is the solution of the RHP $(\tilde{V}^\infty, \Sigma)$, $\tilde{V}^\infty := \begin{pmatrix} \frac{1}{1+s} & 0 \\ 0 & 1+s \end{pmatrix}$. This RHP has the explicit solution

$$\begin{cases} \tilde{M}^\infty = \begin{pmatrix} \frac{1}{1+s} & 0 \\ 0 & 1+s \end{pmatrix}, & |z| < 1, \\ \tilde{M}^\infty = I, & |z| > 1. \end{cases} \tag{48}$$

Therefore as $p \to \infty$, we have the following asymptotics for $z \in \Sigma$,

$$M_+(z;p;s^2) = \tilde{M}_+(z) \begin{pmatrix} 1 & 0 \\ \frac{s}{(1+s)^2} z^p \varphi & 1 \end{pmatrix} \sim \tilde{M}_+^\infty(z) \begin{pmatrix} 1 & 0 \\ \frac{s}{(1+s)^2} z^p \varphi & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{1+s} & 0 \\ \frac{s}{1+s} z^p \varphi & 1+s \end{pmatrix}, \tag{49}$$

where the error is exponentially small, uniformly for z and s.

Now from (38), the above asymptotic result yields for $z \in \Sigma$,

$$\begin{aligned} F(z;p;s^2) &= M_+(z;p;s^2) f(z;p;s^2) \sim \left(\frac{z^{-p}}{1+s}, \frac{-\varphi}{1+s} \right)^T, \\ F'(z;p;s^2) &\sim \left(\frac{-p z^{-p-1}}{1+s}, \frac{-\sqrt{\lambda}(1+z^{-2})\varphi}{1+s} \right)^T, \\ G(z;p;s^2) &= \left(M_+(z;p;s^2)^T \right)^{-1} g(z;p;s^2) \sim \frac{1}{2\pi i} \left(z^p, \varphi^{-1} \right)^T. \end{aligned} \tag{50}$$

Inserting these asymptotics, which have exponential error terms, into (43), equation (42) becomes

$$\begin{aligned} \log\det(I - \sqrt{t}\,\mathrm{K}_p) &= -\int_0^{\sqrt{t}} \frac{ds}{2\pi i(1+s)} \int_\Sigma \left[-p z^{-1} - \sqrt{\lambda}(1+z^{-2}) \right] dz + O\left(e^{-cp}\right) \\ &= p \log(1+\sqrt{t}) + O\left(e^{-cp}\right), \end{aligned} \tag{51}$$

where the exponential error term is uniform for $0 < t \le 1$. □

Combining Lemma 4 and Lemma 5, we summarize the results in this section as follows.

Proposition 6. *Let $m(z;k;t)$ be the solution of the RHP $(v(\cdot\,;t), \Sigma)$ given in* (24)*, and let K_n be defined as in* (16)*. Denoting the 11-component of $m(z;k;t)$ by $m_{11}(z;k;t)$, we have*

$$\prod_{k=n}^{\infty} m_{11}(0;k+1;t) = (1+\sqrt{t})^{-n} \det(I - \sqrt{t}\,\mathrm{K}_n), \qquad n \ge 0. \tag{52}$$

In particular, we have, for the generating function of $l_N^{(1)}$,

$$\phi_n^{(1)}(\lambda) = 2^{-n} \det(I - \mathrm{K}_n), \qquad n \ge 0. \tag{53}$$

Also for $n \ge 0$, the right hand side of (18) *is given by*

$$\phi_n^{(1)}(\lambda) + \left(-\frac{\partial}{\partial t} \right) \bigg|_{t=1} \left[(1+\sqrt{t})^{-n} \det(I - \sqrt{t}\,\mathrm{K}_n) \right] = \left[1 - \sum_{k=n}^{\infty} \frac{\dot{m}_{11}(0;k+1;1)}{m_{11}(0;k+1;1)} \right] \phi_n^{(1)}(\lambda), \tag{54}$$

where the dot $\cdot$ indicates differentiation with respect to t.

Proof. Equation (52) follows from Lemma 4 and Lemma 5.

From (1.25) and (1.27) in [BDJ], we have

$$\phi_n^{(1)}(\lambda) = \prod_{k=n}^{\infty} \big(-Y_{21}(0; k+1)\big), \tag{55}$$

where $Y(z;k)$ solves the RHP (22). From (25), $m_{11}(0;k;1) = -Y_{21}(0;k)$. Hence we have

$$\phi_n^{(1)}(\lambda) = \prod_{k=n}^{\infty} m_{11}(0;k+1) = 2^{-n} \det(I - \mathrm{K}_n). \tag{56}$$

Equation (54) is obtained by taking derivative of (52) and using (53). (The differentiability of the infinite product in (52) follows from the uniform error estimate in the proof of Lemma 5.) ☐

3 Intermediate form of the generating function

For $N, n \geq 0$, let $q_{n,N}^{(2)}$ and $\phi_n^{(2)}(\lambda)$ be defined as in (2) and (3) with $k = 2$. Define

$$\begin{aligned}
a_0(s) &:= \sum_{m=1}^{\infty} \frac{\lambda^m}{(m+s)^2((m-1)!)^2}, \qquad s \geq 1,\\
a_0(0) &:= \sum_{m=0}^{\infty} \frac{\lambda^m}{(m!)^2},\\
a_j(s) &:= \sum_{m=j}^{\infty} \frac{\lambda^{m-j/2}}{(m+s)(m-1)!(m-j)!}, \qquad s \geq 0, j \geq 1\\
b_n(s) &:= \big(a_1(s), \cdots, a_n(s)\big)^T, \qquad s \geq 0.
\end{aligned} \tag{57}$$

Let T_{n-1} be the $n \times n$ Toeplitz matrix given in (14),

$$\mathrm{T}_{n-1} = \left(\int_0^{2\pi} e^{-i(j-k)\theta} e^{2\sqrt{\lambda}\cos\theta} \frac{d\theta}{2\pi} \right)_{0 \leq j,k \leq n-1} \tag{58}$$

Proposition 7. *We have the following expressions for $\phi_{n+1}^{(2)}(\lambda)$,*

$$\phi_{n+1}^{(2)}(\lambda) = \left[\sum_{s=0}^{\infty} \frac{\lambda^s}{(s!)^2} \Big(a_0(s) - \big(T_{n-1}^{-1} b_n(s), b_n(s)\big) \Big) \right] \phi_n^{(1)}(\lambda), \qquad n \geq 1, \tag{59}$$

$$\phi_1^{(2)}(\lambda) = \left[\sum_{s=0}^{\infty} \frac{\lambda^s}{(s!)^2} a_0(s) \right] \phi_0^{(1)}(\lambda). \tag{60}$$

Proof. First note that the statistics of the second row of $\mu \in Y_N$ is the same as the second column of $\mu \in Y_N$. This follows immediately by considering the fact that if π has a k-increasing subsequence, then $\pi'(j) := N + 1 - \pi(j)$ has a k-decreasing subsequence, together with the combinatorial results of Greene referred to in the Introduction relating the rows and columns of a Young tableaux to k-increasing and k-decreasing subsequences. Alternatively, if $\mu \in Y_N$ and if μ' is its transpose, then the hook formula

implies that $d_\mu = d_{\mu'}$, which in turn yields that the k-th row and the k-th column have the same statistics under Plancherel measure.

Let $n \geq 1$. For a partition $\mu \vdash N$, $r_1(\mu)$ and $r_2(\mu)$ are defined to be the lengths of the first and the second columns, respectively. Observe that $\mu = (\mu_1, \mu_2, \cdots, \mu_{r_2(\mu)}, 1, 1, \cdots, 1)$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{r_2(\mu)} \geq 2$. From the remark above, we have

$$q_{n,N}^{(2)} = \sum_{\substack{\mu\vdash N \\ l_N^{(2)}(\mu)\leq n}} \frac{d_\mu^2}{N!} = \sum_{\substack{\mu\vdash N \\ r_2(\mu)\leq n}} \frac{d_\mu^2}{N!} = \sum_{\substack{\mu\vdash N \\ r_1(\mu)\leq n}} \frac{d_\mu^2}{N!} + \sum_{r=n+1}^{N} \sum_{\substack{\mu\vdash N \\ r_1(\mu)=r \\ \mu_{n+1}=\cdots=\mu_r=1}} \frac{d_\mu^2}{N!}. \tag{61}$$

Set $h_j = \mu_j + r_1(\mu) - j$. Then we have the Frobenius-Young determinant formula (see, for example, [Kn] 5.1.4 (34))

$$d_\mu = N! \prod_{1\leq i<j\leq r_1(\mu)} (h_i - h_j) \prod_{j=1}^{r_1(\mu)} \frac{1}{h_j!}, \tag{62}$$

where $\prod_{1\leq i<j\leq r_1(\mu)}(h_i - h_j) := 1$ for $r_1(\mu) = 1$. For $\mu \vdash N$ satisfying $r_1(\mu) = r \geq n+1$ and $\mu_{n+1} = \cdots = \mu_r = 1$, the above expression becomes, after some elementary calculations,

$$d_\mu = \frac{N!}{(r-n)!} \prod_{i=1}^{n} \frac{(h_i-1)!}{(h_i-r+n-1)!} \prod_{1\leq i<j\leq n} (h_i - h_j) \prod_{j=1}^{n} \frac{1}{h_j!}. \tag{63}$$

Then (61) may be re-expressed as

$$q_{n,N}^{(2)} = q_{n,N}^{(1)} \quad + N! \sum_{r=n+1}^{N} \frac{1}{((r-n)!)^2} \sum_{(*)} \prod_{1\leq i<j\leq n} (h_i-h_j)^2 \prod_{j=1}^{n} \frac{1}{(h_j)^2((h_j-r+n-1)!)^2} \tag{64}$$

where $(*)$ means that we sum over all integers $h_1 > h_2 > \cdots > h_n \geq 1+r-n$ such that $\sum_{j=1}^n h_j = N + (r - \frac{1}{2}n)(n-1)$. Of course, the sum in (64) must be replaced by 0 if $n \geq N$. Also, as above, $\prod_{1\leq i<j\leq n}(h_i - h_j)^2 := 1$ if $n = 1$. Hence (3) becomes

$$\begin{aligned} \phi_n^{(2)}(\lambda) &= \phi_n^{(1)}(\lambda) \\ &+ e^{-\lambda} \sum_{r=n+1}^{\infty} \frac{1}{((r-n)!)^2} \sum_{N\geq r} \sum_{(*)} \lambda^N \prod_{1\leq i<j\leq n} (h_i-h_j)^2 \prod_{j=1}^{n} \frac{1}{(h_j)^2(h_j-r+n-1)!)^2}. \end{aligned} \tag{65}$$

Now rewriting $\lambda^N = \lambda^{-(r-\frac{1}{2}n)(n-1)+\sum_{j=1}^n h_j}$ and changing $\sum_{N\geq r}\sum_{(*)}$ into the sum over all integers $h_1 > h_2 > \cdots > h_n \geq 1+r-n$ without any sum restriction, the above expression becomes

$$\begin{aligned} \phi_n^{(2)}(\lambda) &= \phi_n^{(1)}(\lambda) \\ &+ e^{-\lambda} \sum_{r=n+1}^{\infty} \frac{\lambda^{-(r-\frac{1}{2}n)(n-1)}}{((r-n)!)^2} \sum_{h_1>\cdots>h_n\geq 1+r-n} \Big[\prod_{1\leq i<j\leq n} (h_i-h_j)^2 \prod_{j=1}^{n} \frac{\lambda^{h_j}}{(h_j)^2(h_j-r+n-1)!)^2} \Big]. \end{aligned} \tag{66}$$

Changing the summation index to $s := r - n$, setting $l_j := h_j - s$, using the symmetry of the summand under permutation of the h_j's, and noting that the strict ordering of the h_j's is automatically enforced,

we have

$$\phi_n^{(2)}(\lambda) = \phi_n^{(1)}(\lambda) + e^{-\lambda}\lambda^{-n(n-1)/2}\sum_{s=1}^{\infty}\frac{\lambda^s}{(s!)^2}H(s) \tag{67}$$

where

$$H(s) = \frac{1}{n!}\sum_{l_1,\cdots,l_n\geq 1}\Big[\prod_{1\leq i<j\leq n}(l_i-l_j)^2\prod_{j=1}^{n}\frac{\lambda^{l_j}}{(l_j+s)^2((l_j-1)!)^2}\Big], \qquad s\geq 1. \tag{68}$$

Now observe ([Sz], Chapter 2) that $H(s)$ is a Hankel determinant with respect to the discrete measure

$$\nu_s(m) = \frac{\lambda^m}{(m+s)^2((m-1)!)^2}, \qquad m\in\{1,2,\cdots\}, \quad s\geq 1, \tag{69}$$

$$H(s) = \det\Big(\sum_{m=1}^{\infty}m^{j+k}\frac{\lambda^m}{(m+s)^2((m-1)!)^2}\Big)_{0\leq j,k\leq n-1}, \quad s\geq 1. \tag{70}$$

Noting that

$$\int_0^{2\pi}e^{-i(j-k)\theta}e^{2\sqrt{\lambda}\cos\theta}\frac{d\theta}{2\pi} = \sum_{m=\max(j,k)}^{\infty}\frac{\lambda^{m-(j+k)/2}}{(m-j)!(m-k)!}, \tag{71}$$

the identity (13) implies

$$\begin{aligned}\phi_n^{(1)}(\lambda) &= e^{-\lambda}\lambda^{-n(n-1)/2}\det\Big(\sum_{m=0}^{\infty}m^{j+k}\frac{\lambda^m}{(m!)^2}\Big)_{0\leq j,k\leq n-1}\\ &= e^{-\lambda}\lambda^{-n(n-1)/2}H(0),\end{aligned} \tag{72}$$

where $H(0)$ is the Hankel determinant with respect to the discrete measure

$$\nu_0(m) = \frac{\lambda^m}{(m!)^2}, \qquad m\in\{0,1,2,\cdots\}. \tag{73}$$

Therefore we have

$$\phi_n^{(2)}(\lambda) = e^{-\lambda}\lambda^{-n(n-1)/2}\sum_{s=0}^{\infty}\frac{\lambda^s}{(s!)^2}H(s). \tag{74}$$

Using elementary row and column operations, we re-express $H(s)$ as

$$H(s) = \det\big(h(s)_{jk}\big)_{0\leq j,k\leq n-1}, \tag{75}$$

where $h(s)_{jk}$ are defined by

$$\begin{aligned}
h(s)_{00} &= \sum_{m=1}^{\infty}\frac{\lambda^m}{(m+s)^2((m-1)!)^2}, && s\geq 1,\\
h(0)_{00} &= \sum_{m=0}^{\infty}\frac{\lambda^m}{((m!)^2},\\
h(s)_{0k} = h(s)_{k0} &= \sum_{m=k}^{\infty}\frac{\lambda^m}{(m+s)(m-1)!(m-k)!}, && k\geq 1, \quad s\geq 0,\\
h(s)_{jk} &= \sum_{m=\max(j,k)}^{\infty}\frac{\lambda^m}{(m-j)!(m-k)!}, && j,k\geq 1, \quad s\geq 0.
\end{aligned} \tag{76}$$

We obtain

$$\phi_n^{(2)}(\lambda) = e^{-\lambda} \sum_{s=0}^{\infty} \frac{\lambda^s}{(s!)^2} \det\big(\lambda^{-(j+k)/2} h(s)_{jk}\big)_{0 \le j,k \le n-1}. \tag{77}$$

Setting $n = 1$ in (77), we immediately obtain (60). For $n \ge 2$, from (71), $\big(\lambda^{-(j+k)/2} h(s)_{jk}\big)_{1 \le j,k \le n-1}$ is precisely the Toeplitz matrix T_{n-2} in (58). Thus for each s in (77), $\big(\lambda^{-(j+k)/2} h(s)_{jk}\big)_{0 \le j,k \le n-1}$ is a rank 2 extension of T_{n-2}, and hence can be evaluated in the standard way. Indeed expanding the determinant along the first row, and then expanding each of the determinants obtained along the first column, and using (13), $e^{-\lambda} \det(\mathrm{T}_{n-2}) = \phi_{n-1}^{(1)}(\lambda)$, we obtain (59) directly. $\square$

Remarks.

(1) Instead of expanding each determinant in (77) along rows and columns as above, we can appeal directly to the formulae of Weinstein and Aronszajn for the determinant of a finite rank extension of a given operator (see, for example, [Ka] Chapter 4.6).

(2) As we will see in Lemma 10 below, T_{n-1}^{-1}, and hence $\phi_{n+1}^{(2)}(\lambda)$, involves full knowledge of all the monic polynomials $\pi_k(z)$ of degree $k \le n-1$. This is in contrast to $\phi_n^{(1)}(\lambda)$, which involves *only* the leading coefficients κ_m of the normalized orthogonal polynomials $\kappa_m \pi_m(z)$, $m \ge n$, (see [BDJ], (1.25)).

4 Case $n = 0$

In this section, we prove (18) when $n = 0$. From the Proposition 7, it is enough to show that

$$\phi_0^{(1)}(\lambda) + \left(-\frac{\partial}{\partial t}\right)\bigg|_{t=1} \det(I - \sqrt{t}\,\mathrm{K}_0) = \left[\sum_{s=0}^{\infty} \frac{\lambda^s}{(s!)^2} a_0(s)\right] \phi_0^{(1)}(\lambda). \tag{78}$$

But this follows immediately from (17) in the case $n = 0$, and the following Lemma.

Lemma 8. *We have*

$$1 + \frac{1}{2} \operatorname{tr}\left(\frac{1}{I - \mathrm{K}_0} \mathrm{K}_0\right) = \sum_{s=0}^{\infty} \frac{\lambda^s}{(s!)^2} a_0(s). \tag{79}$$

Proof. From (37), (38) and (39), we have

$$\begin{aligned} tr\left(\frac{1}{I - \mathrm{K}_0} \mathrm{K}_0\right) &= \int_{\Sigma} \big(F'(z;0;1)\big)^T G(z;0;1) dz \\ &= \int_{|z|=1-\epsilon} \big(M(z;0;1)^{-1} M'(z;0;1) f(z;0;1)\big)^T g(z;0;1) dz + \int_{|z|=1-\epsilon} \big(f'(z;0;1)\big)^T g(z;0;1) dz, \end{aligned} \tag{80}$$

for any $0 < \epsilon < 1$. From the definition of f, g with $k = 0$ and $t = 1$ in (33), the second integral is 0. We use the relation between M and m in (30), and the relation between m and Y in (25), to express the first integral in terms of $Y(z;0)$. Then we have

$$\operatorname{tr}\left(\frac{1}{I - \mathrm{K}_0} \mathrm{K}_0\right) = \int_{|z|=1-\epsilon} \left(\frac{1}{2} e^{\sqrt{\lambda} z}, -e^{-\sqrt{\lambda} z^{-1}}\right) (Y(z;0)^{-1} Y'(z;0))^T \left(2e^{-\sqrt{\lambda} z}, e^{\sqrt{\lambda} z^{-1}}\right)^T \frac{dz}{2\pi i}. \tag{81}$$

When $n=0$, a simple computation shows that

$$Y(z;0)=\begin{pmatrix}1 & \int_\Sigma \frac{\psi(s)}{s-z}\frac{ds}{2\pi i}\\ 0 & 1\end{pmatrix}. \tag{82}$$

Thus (81) becomes

$$\begin{aligned}\operatorname{tr}\left(\frac{1}{I-\mathrm{K}_0}\mathrm{K}_0\right)&=-2\int_{|z|=1-\epsilon}\psi(z)^{-1}\frac{dz}{2\pi i}\int_\Sigma\frac{\psi(s)}{(s-z)^2}\frac{ds}{2\pi i}\\ &=-2\int_{|z|=1-\epsilon}\frac{dz}{2\pi i}\int_\Sigma\frac{e^{\sqrt{\lambda}(s-z)(1-\frac{1}{sz})}}{(s-z)^2}\frac{ds}{2\pi i}.\end{aligned} \tag{83}$$

Now by Taylor expansion,

$$\frac{e^{\sqrt{\lambda}(s-z)(1-\frac{1}{sz})}}{(s-z)^2}=\sum_{k=0}^\infty\frac{(\sqrt{\lambda})^k}{k!}(s-z)^{k-2}\big(1-\frac{1}{sz}\big)^k=\sum_{k=0}^\infty\frac{(\sqrt{\lambda})^k}{k!}\sum_{l=0}^k\binom{k}{l}(s-z)^{k-2}\frac{(-1)^l}{s^lz^l}. \tag{84}$$

For $|z|=1-\epsilon$, we have

$$\int_\Sigma\frac{e^{\sqrt{\lambda}(s-z)(1-\frac{1}{sz})}}{(s-z)^2}\frac{ds}{2\pi i}=\sqrt{\lambda}-\sum_{k=2}^\infty\frac{(-\sqrt{\lambda})^k}{k!}\sum_{l=1}^{k-1}\binom{k}{l}\binom{k-2}{l-1}z^{k-2l-1}. \tag{85}$$

Therefore

$$\operatorname{tr}\left(\frac{1}{I-\mathrm{K}_0}\mathrm{K}_0\right)=2\sum_{k\geq 2,even}\frac{(-\sqrt{\lambda})^k}{k!}\binom{k}{k/2}\binom{k-2}{k/2-1}=2\sum_{p=1}^\infty\frac{\lambda^p}{(2p)!}\binom{2p}{p}\binom{2p-2}{p-1}. \tag{86}$$

Thus we have

$$1+\frac{1}{2}\operatorname{tr}\left(\frac{1}{I-\mathrm{K}_0}\mathrm{K}_0\right)=1+\sum_{p=1}^\infty\frac{\lambda^p}{(p!)^2}\binom{2p-2}{p-1}. \tag{87}$$

On the other hand, from the definition of $a_0(s)$ in (57), the right hand side of (79) is given by

$$\sum_{s=0}^\infty\frac{\lambda^s}{(s!)^2}a_0(s)=1+\sum_{s=0}^\infty\frac{\lambda^s}{(s!)^2}\sum_{m=1}^\infty\frac{\lambda^m}{(m+s)^2((m-1)!)^2}. \tag{88}$$

Changing the summation index to $p:=s+m$, this expression becomes

$$1+\sum_{p=1}^\infty\sum_{s=0}^{p-1}\binom{p-1}{s}^2\frac{\lambda^p}{(p!)^2}=1+\sum_{p=1}^\infty\binom{2p-2}{p-1}\frac{\lambda^p}{(p!)^2}, \tag{89}$$

which agrees with (87). Here we have used the elementary combinatorial identity

$$\sum_{s=0}^{p-1}\binom{p-1}{s}^2=\binom{2p-2}{p-1}. \tag{90}$$

This proves the Lemma. $\square$

5 Case $n > 0$

From (54) and (59), we need to verify that

$$1 - \sum_{k=n}^{\infty} \frac{\dot{m}_{11}(0;k+1;1)}{m_{11}(0;k+1;1)} = \sum_{s=0}^{\infty} \frac{\lambda^s}{(s!)^2} \Big(a_0(s) - \big(T_{n-1}^{-1} b_n(s), b_n(s) \big) \Big). \tag{91}$$

But using (54) and (78), it is enough to show

$$-\sum_{k=0}^{n-1} \frac{\dot{m}_{11}(0;k+1;1)}{m_{11}(0;k+1;1)} = \sum_{s=0}^{\infty} \frac{\lambda^s}{(s!)^2} \big(T_{n-1}^{-1} b_n(s), b_n(s) \big). \tag{92}$$

To this end, we need the following two Lemmas. Recall $\psi(z) := e^{\sqrt{\lambda}(z+z^{-1})}$.

Lemma 9. *Let $0 < \epsilon < 1$. For $p, q \geq 0$, we have*

$$\sum_{s=0}^{\infty} \frac{\lambda^s}{(s!)^2} a_{p+1}(s) a_{q+1}(s) = -\int_{|z|=1-\epsilon} \psi(-z) \left(\int_{|u|=1} \frac{u^q \psi(u)}{u-z} \frac{du}{2\pi i} \right) \left(\int_{|v|=1} \frac{\psi(v)}{v^{p+1}(v-z)} \frac{dv}{2\pi i} \right) \frac{dz}{2\pi i}. \tag{93}$$

Proof. From the definition of $a_k(s)$, the left hand side of (93) is

$$\begin{aligned} &\sum_{s=0}^{\infty} \frac{\lambda^s}{(s!)^2} \sum_{a=p+1}^{\infty} \frac{\lambda^{a-(p+1)/2}}{(a+s)(a-1)!(a-p-1)!} \sum_{b=q+1}^{\infty} \frac{\lambda^{b-(q+1)/2}}{(b+s)(b-1)!(b-q-1)!} \\ &= \lambda^{-\frac{p+q}{2}} \sum_{s=0}^{\infty} \sum_{a=p+1}^{\infty} \sum_{b=q+1}^{\infty} \frac{\lambda^{s+a+b-1}}{(s+a)!(s+b)!(a-p-1)!(b-q-1)!} \binom{s+a-1}{a-1} \binom{s+b-1}{b-1}. \end{aligned}$$

Setting $k = s+a+b-1-p-q$, $l = a-p$ and $m = b-q$, the above expression becomes

$$\lambda^{\frac{p+q}{2}} \sum_{k=1}^{\infty} \lambda^k \sum_{l=1}^{k} \sum_{m=1}^{k-l+1} \frac{1}{(k-m+p+1)!(k-l+q+1)!(l-1)!(m-1)!} \binom{k-m+p}{l+p-1} \binom{k-l+q}{m+q-1}. \tag{94}$$

On the other hand, the right hand side of (93) is

$$-\int_{|z|=1-\epsilon} \frac{dz}{2\pi i} \int_{|u|=1} \frac{u^q e^{\sqrt{\lambda} u} e^{\sqrt{\lambda} \frac{(z-u)}{uz}}}{u-z} \frac{du}{2\pi i} \int_{|v|=1} \frac{e^{\sqrt{\lambda} v^{-1}} e^{\sqrt{\lambda}(v-z)}}{v^{p+1}(v-z)} \frac{dv}{2\pi i}. \tag{95}$$

For the third integral in (95), Taylor expansions of $e^{\sqrt{\lambda}(v-z)}$ and $e^{\sqrt{\lambda} v^{-1}}$ give us

$$\sum_{a=1}^{\infty} \frac{(\sqrt{\lambda})^a}{a!} \sum_{b=0}^{\infty} \frac{(\sqrt{\lambda})^b}{b!} \int_{|v|=1} \frac{(v-z)^{a-1}}{v^{b+p+1}} \frac{dv}{2\pi i},$$

since the integration when $a = 0$ vanishes. Evaluating the integral, we obtain

$$\sum_{a=p+1}^{\infty} \frac{(\sqrt{\lambda})^a}{a!} \sum_{b=0}^{a-p-1} \frac{(\sqrt{\lambda})^b}{b!} \binom{a-1}{b+p} (-z)^{a-b-p-1}. \tag{96}$$

Similarly by expanding $e^{\sqrt{\lambda} \frac{(z-u)}{uz}}$ and $e^{\sqrt{\lambda} u}$ in Taylor series, the second integral in (95) becomes

$$z^q e^{\sqrt{\lambda} z} + \sum_{c=q+1}^{\infty} \frac{(\sqrt{\lambda})^c}{c!} \sum_{d=0}^{c-q-1} \frac{(\sqrt{\lambda})^d}{d!} \binom{c-1}{d+q} (-z)^{-c+d+q}. \tag{97}$$

Now using (96) and (97), (95) becomes

$$\sum_{a=p+1}^{\infty}\sum_{b=0}^{a-p-1}\sum_{c=q+1}^{\infty}\sum_{d=0}^{c-q-1}\frac{(-1)^{q-p}(-\sqrt{\lambda})^{(a+b+c+d)}}{a!b!c!d!}\binom{a-1}{b+p}\binom{c-1}{d+q}\int_{|z|=1-\epsilon} z^{a-b-c+d-p+q-1}\frac{dz}{2\pi i} \tag{98}$$

since the contribution from $z^q e^{\sqrt{\lambda}z}$ is zero. The integral in (98) is 1 when $d=-a+b+c+p-q$, which gives a new restriction on c, $c\geq a-b-p+q$, implying

$$\sum_{a=p+1}^{\infty}\sum_{b=0}^{a-p-1}\sum_{c=a-b-p+q}^{\infty}\frac{\lambda^{(b+c+(p-q)/2)}}{a!b!c!(-a+b+c+p-q)!}\binom{a-1}{b+p}\binom{c-1}{-a+b+c+p}.$$

Changing the summation indices to k,l,m by $b=l-1$, $c+b-q=k$ and $-a+b+c+p-q+1=m$, the above expression is (94). $\square$

Lemma 10. *Let*

$$\pi_n(z)=z^n+\cdots=\sum_{p=0}^{n}\eta_p^n z^p, \qquad \eta_n^n=1 \tag{99}$$

be the n-th monic orthogonal polynomial with respect to a measure $f(e^{i\theta})d\theta/2\pi$ on the unit circle which satisfies $f(e^{i\theta})=f(e^{-i\theta})$. Let $\mathrm{T}_n=(c_{j-k})_{0\leq j,k\leq n}$ denote the $(n+1)\times(n+1)$ Toeplitz matrix with respect to the same measure. Then we have for $n\geq 1$,

$$\begin{aligned}\left(\mathrm{T}_n^{-1}\right)_{pq} &= \kappa_n^2\eta_p^n\eta_q^n, && p=n \ \ \textit{or} \ \ q=n, && (100)\\ \left(\mathrm{T}_n^{-1}\right)_{pq}-\left(\mathrm{T}_{n-1}^{-1}\right)_{pq} &= \kappa_n^2\eta_p^n\eta_q^n, && 0\leq p,q\leq n-1, && (101)\end{aligned}$$

where κ_n is the leading coefficient of the n-th normalized orthogonal polynomial, $\int_0^{2\pi}|\kappa_n\pi_n(e^{i\theta})|^2 f(e^{i\theta})d\theta/2\pi=1$.

Proof. Set

$$\gamma_{pq}:=\begin{cases}\left(\mathrm{T}_n^{-1}\right)_{pq}, & p=n \ \text{ or } \ q=n,\\ \left(\mathrm{T}_n^{-1}\right)_{pq}-\left(\mathrm{T}_{n-1}^{-1}\right)_{pq}, & 0\leq p,q\leq n-1.\end{cases} \tag{102}$$

Define

$$b(z,w):=\sum_{p,q=0}^{n}\gamma_{pq}z^p w^q. \tag{103}$$

Using the definition of the Toeplitz coefficient $c_k=\int_0^{2\pi}e^{-ik\theta}f(e^{i\theta})d\theta/2\pi$,

$$\begin{aligned}\int_0^{2\pi} b(e^{i\theta},w)e^{-ij\theta}f(e^{i\theta})\frac{d\theta}{2\pi} &= \sum_{p,q=0}^{n}\gamma_{pq}c_{j-p}w^q\\ &= \sum_{p,q=0}^{n}\left(\mathrm{T}_n^{-1}\right)_{pq}c_{j-p}w^q-\sum_{p,q=0}^{n-1}\left(\mathrm{T}_{n-1}^{-1}\right)_{pq}c_{j-p}w^q.\end{aligned} \tag{104}$$

But $c_{j-p} = \left(\mathrm{T}_n\right)_{jp}$ for $0 \le j,p \le n$ and also $c_{j-p} = \left(\mathrm{T}_{n-1}\right)_{jp}$ for $0 \le j,p \le n-1$. Therefore (104) is zero for $0 \le j \le n-1$. This shows that for fixed w, $b(z,w)$ is a polynomial in z of degree n which is orthogonal to $1, z, \cdots, z^{n-1}$. Thus, for some $a(w)$,

$$b(z,w) = \pi_n(z)a(w). \tag{105}$$

Now the evenness of f, $f(e^{i\theta}) = f(e^{-i\theta})$, implies that $c_k = c_{-k}$, and hence the Toeplitz matrices above are symmetric, $\gamma_{pq} = \gamma_{qp}$, and so $b(z,w) = b(w,z)$. Thus

$$b(z,w) = c\pi_n(z)\pi_n(w), \tag{106}$$

for some constant c. To determine the constant c, we consider the coefficient of the leading term $z^n w^n$ of $b(z,w)$. That is

$$\gamma_{nn} = \left(\mathrm{T}_n^{-1}\right)_{nn} = \frac{\det(\mathrm{T}_{n-1})}{\det(\mathrm{T}_n)} = \kappa_n^2, \tag{107}$$

by [Sz]. Thus we have

$$b(z,w) = \kappa_n^2 \pi_n(z)\pi_n(w) \tag{108}$$

and this completes the proof. $\square$

Finally we prove (92) which in turn completes the proof of Theorem 1. Let $m(z;k;t)$ be the solution of the RHP $(v(z;t),\Sigma)$ given in (24).

Lemma 11. *We have for all* $n \ge 1$,

$$-\sum_{k=0}^{n-1} \frac{\dot{m}_{11}(0;k+1;1)}{m_{11}(0;k+1;1)} = \sum_{s=0}^{\infty} \frac{\lambda^s}{(s!)^2} \left(\mathrm{T}_{n-1}^{-1}\, b_n(s), b_n(s)\right), \tag{109}$$

Proof. In the proof that follows, m,$\dot{m}$,v,$\dot{v}$ are all evaluated at $t = 1$. By differentiating the RHP (24) with respect to t, we have

$$\begin{cases} \dot{m}_+ = \dot{m}_- v + m_- \dot{v}, & \text{on } \Sigma, \\ \dot{m} = O(1/z) & \text{as } z \to \infty. \end{cases} \tag{110}$$

Since m satisfies $m_+ = m_- v$, we have

$$\begin{cases} \left(\dot{m}m^{-1}\right)_+ = \left(\dot{m}m^{-1}\right)_- + m_-\dot{v}v^{-1}m_-^{-1} = \left(\dot{m}m^{-1}\right)_- + m_+ v^{-1}\dot{v}m_+^{-1}, & \text{on } \Sigma, \\ \dot{m}m^{-1} = O(1/z) & \text{as } z \to \infty. \end{cases} \tag{111}$$

From the Plemelj formula, the solution of this equation is given by

$$\left(\dot{m}m^{-1}\right)(z) = \int_\Sigma \frac{\left(m_+ v^{-1}\dot{v}m_+^{-1}\right)(s)}{s-z} \frac{ds}{2\pi i}. \tag{112}$$

Therefore for any $0 < \epsilon < 1$, we have

$$\dot{m}_{11}(0) = \left[\int_{|z|=1-\epsilon} m(z)v^{-1}(z)\dot{v}(z)m^{-1}(z)m(0)\frac{dz}{2\pi iz}\right]_{11}. \tag{113}$$

To simplify the calculation note that the symmetry of the jump matrix $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)v(1/z)\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) = v(z)^{-1}$ implies the symmetry of the solution

$$m(0)^{-1}m(1/z) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} m(z) \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{114}$$

Using this relation for $m^{-1}(z)m(0)$ in (113), we have

$$\dot{m}_{11}(0) = \left[\int_{|z|=1-\epsilon} m(z)v^{-1}(z)\dot{v}(z)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} m^{-1}(1/z)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{dz}{2\pi iz}\right]_{11}. \tag{115}$$

Note that $\dot{v}(z;k+1;1) = \left(\begin{smallmatrix}-1 & -1/2z^{-k-1}\varphi^{-1}\\ 1/2z^{k+1}\varphi & 0\end{smallmatrix}\right)$. Now we express (115) in terms of Y using the relation (25),

$$\begin{aligned}\dot{m}_{11}(0) = \int_{|z|=1-\epsilon}\Big[&\frac{1}{2}\psi(z)Y_{21}(z)Y_{21}(1/z) + \frac{1}{2}z^{-k-1}Y_{21}(z)Y_{22}(1/z)\\ &-\frac{1}{2}z^{k+1}Y_{22}(z)Y_{21}(1/z) - \psi(z)^{-1}Y_{22}(z)Y_{22}(1/z)\Big]\frac{dz}{2\pi iz},\end{aligned} \tag{116}$$

where $\psi(z) = e^{\sqrt{\lambda}(z+z^{-1})}$ as given in (21).

Now we use the explicit expression of Y in terms of orthogonal polynomial given in (23). Especially we use the following expressions,

$$\begin{aligned}Y_{21}(z) &= -\kappa_k^2 z^k \pi_k(1/z), &(117)\\ Y_{21}(1/z) &= -\kappa_k^2 z^{-k}\pi_k(z), &(118)\\ Y_{22}(z) &= -\kappa_k^2\int_{|v|=1}\frac{\pi_k(1/v)\psi(v)}{v(v-z)}\frac{dv}{2\pi i}, &(119)\\ Y_{22}(1/z) &= \kappa_k^2\int_{|u|=1}\frac{z\pi_k(u)\psi(u)}{u-z}\frac{du}{2\pi i}. &(120)\end{aligned}$$

The first two terms in (116) cancel each other, which can be seen as follows :

$$\begin{aligned}\text{first term} &= \frac{\kappa_k^4}{2}\int_{|z|=1-\epsilon}\psi(z)\pi_k(1/z)\pi_k(z)\frac{dz}{2\pi iz},\\ \text{second term} &= \frac{-\kappa_k^4}{2}\int_{|z|=1-\epsilon}\pi_k(1/z)\left(\int_{|u|=1}\frac{\pi_k(u)\psi(u)}{u-z}\frac{du}{2\pi i}\right)\frac{dz}{2\pi iz}\\ &= \frac{-\kappa_k^4}{2}\int_{|u|=1}\pi_k(u)\psi(u)\frac{du}{2\pi i}\int_{|z|=1-\epsilon}\frac{\pi_k(1/z)}{u-z}\frac{dz}{2\pi iz}\\ &= \frac{-\kappa_k^4}{2}\int_{|u|=1}\pi_k(u)\psi(u)\pi_k(1/u)\frac{du}{2\pi iu},\end{aligned} \tag{121}$$

which is -1 times the first term, by Cauchy. Also the third term in (116) vanishes as

$$\begin{aligned}\text{third term} &= \frac{-\kappa_k^4}{2}\int_{|z|=1-\epsilon}\pi_k(z)\left(\int_{|v|=1}\frac{\pi_k(1/v)\psi(v)}{v(v-z)}\frac{dv}{2\pi i}\right)\frac{dz}{2\pi i}\\ &= \frac{-\kappa_k^4}{2}\int_{|v|=1}\pi_k(1/v)\psi(v)\frac{dv}{2\pi i v}\int_{|z|=1-\epsilon}\frac{\pi_k(z)}{v-z}\frac{dz}{2\pi i}=0,\end{aligned} \tag{122}$$

since the quantity $\frac{\pi_k(z)}{v-z}$ is analytic for $|z|\le 1-\epsilon$. Thus together with the fact $m_{11}(0;k+1;1)=\kappa_k^2$, which follows immediately from (23) and (25), we have

$$-\frac{\dot m_{11}(0;k+1;1)}{m_{11}(0;k+1;1)} = -\kappa_k^2\int_{|z|=1-\epsilon}\psi(z)^{-1}\left(\int_{|u|=1}\frac{\pi_k(u)\psi(u)}{u-z}\frac{du}{2\pi i}\right)\left(\int_{|v|=1}\frac{\pi_k(1/v)\psi(v)}{v(v-z)}\frac{dv}{2\pi i}\right)\frac{dz}{2\pi i}. \tag{123}$$

As in (99), let

$$\pi_k(u)=\sum_{q=0}^{k}\eta_q^k u^q, \qquad \pi_k(1/v)=\sum_{p=0}^{k}\eta_p^k v^{-p}. \tag{124}$$

Then (123) becomes

$$-\frac{\dot m_{11}(0;k+1;1)}{m_{11}(0;k+1;1)} = -\kappa_k^2\sum_{p,q=0}^{k}\eta_q^k\eta_p^k\int_{|z|=1-\epsilon}\psi(z)^{-1}\left(\int_{|u|=1}\frac{u^q\psi(u)}{u-z}\frac{du}{2\pi i}\right)\left(\int_{|v|=1}\frac{\psi(v)}{v^{p+1}(v-z)}\frac{dv}{2\pi i}\right)\frac{dz}{2\pi i}, \tag{125}$$

and hence by Lemma 9,

$$-\frac{\dot m_{11}(0;k+1;1)}{m_{11}(0;k+1;1)} = \sum_{s=0}^{\infty}\frac{\lambda^s}{(s!)^2}\sum_{p,q=0}^{k}\kappa_k^2\eta_p^k\eta_q^k a_{p+1}(s)a_{q+1}(s). \tag{126}$$

But then from Lemma 10, we have for $k\ge 1$,

$$-\frac{\dot m_{11}(0;k+1;1)}{m_{11}(0;k+1;1)} = \sum_{s=0}^{\infty}\frac{\lambda^s}{(s!)^2}\left[\sum_{p,q=0}^{k}\left(\mathrm{T}_k^{-1}\right)_{pq}a_{p+1}(s)a_{q+1}(s) - \sum_{p,q=0}^{k-1}\left(\mathrm{T}_{k-1}^{-1}\right)_{pq}a_{p+1}(s)a_{q+1}(s)\right]. \tag{127}$$

For $k=0$, $\eta_0^0=1$, T_0 is the 1×1 matrix with entry κ_0^{-2}, and so by (126),

$$-\frac{\dot m_{11}(0;1;1)}{m_{11}(0;1;1)} = \sum_{s=0}^{\infty}\frac{\lambda^2}{(s!)^2}\left(\mathrm{T}_0^{-1}\right)a_1(s)a_1(s). \tag{128}$$

Thus we have

$$-\sum_{k=0}^{n-1}\frac{\dot m_{11}(0;k+1;1)}{m_{11}(0;k+1;1)} = \sum_{s=0}^{\infty}\frac{\lambda^s}{(s!)^2}\sum_{p,q=0}^{n-1}\left(\mathrm{T}_{n-1}^{-1}\right)_{pq}a_{p+1}(s)a_{q+1}(s) = \sum_{s=0}^{\infty}\frac{\lambda^s}{(s!)^2}\left(\mathrm{T}_{n-1}^{-1}\,b_n(s),b_n(s)\right). \tag{129}$$

$\square$

6 Asymptotics

In this section, we make some remarks concerning the proofs of Theorem 2 and 3.

First, as in [Jo], one can show that $q_{n,N}^{(2)}$ is monotonically decreasing in N, and so by the de-Poissonization Lemma (see Lemma 2.5 in [Jo]), it is enough to control $\phi_n^{(2)}(\lambda)$ as $n,\lambda \to \infty$. By (18), this translates into controlling $\det(I - \sqrt{t}\,\mathrm{K}_n)$ for values of t near 1 (of course, the asymptotic behavior of $\phi_n^{(1)}(\lambda)$ is given in [BDJ]). But as noted in the Introduction, K_n, and hence $\sqrt{t}\,\mathrm{K}_n$, is an integrable operator, to which there is an canonically associated RHP (see (31)). As in [BDJ], the steepest descent method can be used to analyze this RHP asymptotically as $\lambda, n \to \infty$. Again the critical region is where $n \sim 2\sqrt{\lambda}$, in which case the RHP localizes to a small neighborhood of $z = -1$. Write $2\sqrt{\lambda} = k - xk^{1/3}/2^{1/3}$, where x lies in a bounded set. Writing $z = -1 + s$ for z near -1, we obtain

$$\begin{aligned} v(z;k;t) &= \begin{pmatrix} 1-t & -\sqrt{t}z^{-k}e^{-\sqrt{\lambda}(z-z^{-1})} \\ \sqrt{t}z^{k}e^{\sqrt{\lambda}(z-z^{-1})} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1-t & -\sqrt{t}z^{-k}(-1)^k e^{-h(s,k,x)} \\ \sqrt{t}z^{k}(-1)^k e^{h(s,k,x)} & 1 \end{pmatrix}, \end{aligned} \tag{130}$$

where

$$\begin{aligned} h(s,k,x) &= -\frac{x}{2^{1/3}}(k^{1/3}s) - \frac{x}{2^{4/3}k^{1/3}}(k^{1/3}s)^2 + \frac{(k^{1/3}s)^3}{6}\left(1 - \frac{3x}{k^{2/3}2^{1/3}}\right) + \cdots \\ &\sim -\frac{x}{2^{1/3}}(k^{1/3}s) + \frac{(k^{1/3}s)^3}{6}, \qquad \text{as } k \to \infty. \end{aligned} \tag{131}$$

Rescaling $\tilde{s} = k^{1/3}s/2^{4/3}$, we see that we are lead to a RHP with jump matrix

$$\tilde{v} = \begin{pmatrix} 1-t & -(-1)^k\sqrt{t}e^{-2(-x\tilde{s}+\frac{4}{3}\tilde{s}^3)} \\ (-1)^k\sqrt{t}e^{2(-x\tilde{s}+\frac{4}{3}\tilde{s}^3)} & 1 \end{pmatrix} \tag{132}$$

on the line $i\mathbb{R}$ (cf. Figure 9 in [BDJ]). But after rotating by $\pi/2$, this is precisely the RHP for the Painlevé II equation with parameters $p = -q = \sqrt{t}$, $r = 0$ (cf. Figure 4 in [BDJ] : the terms $(-1)^k$ can be removed by a simple conjugation). These parameters p, q, r correspond to the solution $u(x;t)$ of the Painlevé II equation, $u_{xx} = 2u^3 + xu$, with the boundary condition $u(x;t) \sim -\sqrt{t}Ai(x)$ as $x \to +\infty$, where Ai is the Airy function (cf. [BDJ] (1.4)). As in Lemmas 5.1 and 6.3 in [BDJ], we can obtain an expression for $m_{11}(0;k+1;t)$ in terms of the solution of the above Painlevé II RHP. Inserting this information into (52) in Proposition 6, we learn that for $2\sqrt{\lambda} = n - xn^{1/3}/2^{1/3}$, $(1+\sqrt{t})^{-n}\det(I - \sqrt{t}\,\mathrm{K}_n) \to F(x;t)$ as $n \to \infty$. Substituting this relation into (18), we obtain the proof of Theorem 2, and eventually Theorem 3.

- -

We conclude with some remarks on the motivation for a formula such as (18). We started backwards, assuming that the second row behaves statistically in the large N limit like the second largest eigenvalue of a GUE random matrix. As noted in the Introduction, this conjecture was strongly supported by

numerical simulations of Odlyzko and Rains. We had to end up with the Tracy-Widom distribution $F^{(2)}(x)$. From the point of view of [BDJ], $F^{(2)}(x)$ would have to emerge from the solution of some local RHP. For the case $F(x;1)$, which is expressed (see (6)) in terms of a specific solution $u(x;1)$ of the Painlevé II equation, $u_{xx} = 2u^3 + xu$, $u(x;1) \sim -Ai(x)$ as $x \to +\infty$, the local RHP was precisely the RHP for the Painlevé II equation, and this local problem emerged naturally via the steepest descent method applied to the RHP associated to Gessel's formula (13) in the canonical way. But now $F^{(2)}$ is a derivative of $F(x;t)$ where $F(x;t)$ involves a family of solutions $\{u(x;t)\}$ of the Painlevé II equation, $u_{xx} = 2u^3+xu$, $u(x;t) \sim -\sqrt{t}Ai(x)$ as $x \to +\infty$. So we need to find a RHP which reduces in the critical region $n \sim 2\sqrt{\lambda}$ to a local RHP, which is precisely the RHP for the solution $u(x;t)$ of the Painlevé II equation. The RHP (24) is chosen precisely to ensure this property.

The procedure leading to (18) is now forced. The RHP (24) (more precisely, the equivalent RHP (31)) is of the type that arises from an integrable operator (K_n in this case), which then leads after some calculations to the determinant formula for $I - \mathrm{K}_n$ on the right hand side of (18).

Appendix

In this Appendix, we first discuss the spectral properties of the operator K_n in (16) that are used in the proof of Lemma 4, and then the (unique) solvability of the RHP (24).

Let Σ denote the unit circle in the complex plane, oriented counterclockwise, and let $\varphi(z) = e^{\sqrt{\lambda}(z-z^{-1})}$, as before. The operator $\mathrm{K}_n : L^2(\Sigma, |dz|) \to L^2(\Sigma, |dz|)$ is defined by

$$\mathrm{K}_n(z,w) := \frac{z^{-n}w^n - \varphi(z)\varphi(w)^{-1}}{2\pi i(z-w)}, \qquad \big(\mathrm{K}_n f\big)(z) = \int_\Sigma \mathrm{K}_n(z,w)f(w)dw. \tag{133}$$

First note that

$$\mathrm{K}_n = \frac{1}{2}A\widehat{H}B, \tag{134}$$

where the operators $A : L^2(\Sigma,|dz|) \oplus L^2(\Sigma,|dz|) \to L^2(\Sigma,|dz|)$ and $B : L^2(\Sigma,|dz|) \to L^2(\Sigma,|dz|) \oplus L^2(\Sigma,|dz|)$ are defined by

$$\big(A\vec{h}\big)(z) := z^{-n}h_1(z) + \varphi(z)h_2(z), \qquad \big(Bh\big)(z) := \big(-z^n h(z), \varphi(z)^{-1}h(z)\big)^T, \tag{135}$$

for a scalar h and a vector $\vec{h} = (h_1,h_2)^T$, and $\widehat{H} : L^2(\Sigma,|dz|) \oplus L^2(\Sigma,|dz|) \to L^2(\Sigma,|dz|) \oplus L^2(\Sigma,|dz|)$ is defined by

$$\big(\widehat{H}\vec{h}\big)(z) := \big((Hh_1)(z), (Hh_2)(z)\big)^T, \tag{136}$$

for a vector $\vec{h} = (h_1,h_2)^T$ where $H : L^2(\Sigma,|dz|) \to L^2(\Sigma,|dz|)$ is the Hilbert transformation given by

$$\big(Hh\big)(z) = \lim_{\epsilon\to 0}\frac{1}{i\pi}\int_{|s|=1,|s-z|>\epsilon}\frac{h(s)}{s-z}ds. \tag{137}$$

Since $\|A\| \le \sqrt{2}$, $\|Bh\| = \sqrt{2}\|h\|$ and $\|Hh\| = \|h\|$, we have $\|\,\mathrm{K}_n\,\| \le 1$.

As $\varphi(z) = \overline{\varphi(z)^{-1}}$, we have

$$\mathrm{K}_n(z,w)dw = \frac{\overline{z^n}w^n - \overline{\varphi(z)^{-1}}\varphi(w)^{-1}}{2\pi\big(\overline{z^{-1}}w^{-1} - 1\big)}\frac{dw}{iw}. \tag{138}$$

Since $\frac{dw}{iw} = d\theta = |dw|$, K_N is a self-adjoint operator on $L^2(\Sigma, |dz|)$. Also since the kernel is smooth, the operator K_n is trace class, and hence $\|\,\mathrm{K}_n\,\| = 1$ if and only if $+1$ and/or -1 is an eigenvalue.

We show that 1 is not an eigenvalue of K_n. Observe first that

$$\|A\vec{h}\| = \sqrt{2}\|\vec{h}\| \quad \text{if and only if} \quad z^{-n}h_1(z) = \varphi(z)h_2(z). \tag{139}$$

Now suppose that 1 is an eigenvalue of K_n. Then there is a non-trivial function $h \in L^2(\Sigma, |dz|)$ such that $\mathrm{K}_n\, h = h$. Then

$$\|h\| = \|K_n h\| = \frac{1}{2}\|A\widehat{H}Bh\| \leq \frac{1}{\sqrt{2}}\|\widehat{H}Bh\| = \|h\|, \tag{140}$$

which implies that

$$\|A\widehat{H}Bh\| = \sqrt{2}\|\widehat{H}Bh\|. \tag{141}$$

Hence by (139) above, and by the definition of the operator B given in (135), we have an equation

$$-z^{-n}H(z^n h) = \varphi(z)H(\varphi(z)^{-1}h). \tag{142}$$

Now re-express K_n as follows,

$$\mathrm{K}_n\, h = -\frac{1}{2}z^{-n}H(z^n h) + \frac{1}{2}\varphi(z)H(\varphi(z)^{-1}h). \tag{143}$$

Using (142),

$$\mathrm{K}_n\, h = -z^{-n}H(z^n h) = \varphi(z)H(\varphi(z)^{-1}h), \tag{144}$$

which leads to the equations

$$-z^{-n}H(z^n h) = h, \qquad \varphi(z)H(\varphi(z)^{-1}h) = h, \tag{145}$$

or

$$H(z^n h) = -z^n h, \qquad H(\varphi(z)^{-1}h) = \varphi(z)^{-1}h. \tag{146}$$

But $H(z^n) = z^n$ for $n \geq 0$ and $H(z^n) = -z^n$ for $n < 0$, so that (146) implies

$$z^n h(z) = \sum_{j<0} a_j z^j, \qquad \varphi(z)^{-1}h(z) = \sum_{j\geq 0} b_j z^j, \tag{147}$$

for some square summable sequences $\{a_j\}_{j<0}$ and $\{b_j\}_{j\geq 0}$. Hence, the second equation in (147) implies that

$$e^{\sqrt{\lambda}z^{-1}}h(z) = e^{\sqrt{\lambda}z}\sum_{j\geq 0} b_j z^j. \tag{148}$$

Combining with the first equation in (147),

$$e^{\sqrt{\lambda}z^{-1}}\sum_{j<0}a_jz^{j-n}=e^{\sqrt{\lambda}z}\sum_{j\geq 0}b_jz^j, \tag{149}$$

which is impossible for $n\geq 0$ unless all the a_j's (and b_j's) are zero. Therefore for $n\geq 0$, 1 is not an eigenvalue of K_n. In a similar manner, one can show that $\dim Ker(\mathrm{K}_n+1)=n$, for $n\geq 0$. In particular, $\|\mathrm{K}_n\|=1$ and $\dim Ker(\mathrm{K}_n-1)=0$ for $n\geq 0$.

Now we prove the (unique) solvability of the RHP (24). It is clear that the solvability of the RHP (24) follows from the solvability of the RHP (31) since m and M are algebraically related by (30). Now from integrable operator theory (see Lemma 2.21 [DIZ]), the existence of the inverse of $(I-\sqrt{t}\,\mathrm{K}_k)^{-1}$ implies the solvability of the RHP (31). But as $\|K_k\|=1$, and as 1 is not in the spectrum of K_k, it follows that $(I-\sqrt{t}\,\mathrm{K}_k)^{-1}$ exit for all $0<t\leq 1$, and hence the RHP (24) is solvable. The proof of the uniqueness of the solution of the RHP is standard (compare, for example, [BDJ] Lemma 4.2).

Remark. Note from (17) that when $\lambda=0$, $\det(I-\mathrm{K}_n(\lambda=0))=2^n\phi_n^{(1)}(0)=2^n$, which can be checked directly (for $\lambda=0$, $-\mathrm{K}_n(\lambda=0)$ is an orthogonal projection of rank n). On the other hand, as $\lambda,n\to\infty$, the spectrum of $\mathrm{K}_n=\mathrm{K}_n(\lambda)$ can approach 1 : indeed from Lemma 7.1 (v) in [BDJ], for $2\sqrt{\lambda}\geq(n+1)(1+\delta_7)\to\infty$, $\delta_7>0$, $\phi_n^{(1)}(\lambda)\leq Ce^{-cn^2}$ so that $\det(I-\mathrm{K}_n)=2^n\phi_n^{(1)}(\lambda)\to 0$ as $n\to\infty$.

References

[BDJ] J.Baik, P.Deift and K.Johansson, *On the Distribution of the Length of the Longest Increasing Subsequence of Random Permutations*, preprint, 1998.

[CG] K.Clancey and I.Gohberg, *Factorization of Matrix Functions and Singular Integral Operators*, Birkhäuser, 1981.

[De] P.A.Deift, *Integrable Operators*, preprint, 1998.

[DIZ] P.A.Deift, A.R.Its and X.Zhou, *A Riemann-Hilbert approach to asymptotic problems arising in the theory of random matrix models, and also in the theory of integrable statistical mechanics*, Ann. Math., **146**, 149-235, (1997).

[DKMVZ] P.A.Deift, T.Kriecherbauer, K.T-R McLaughlin, S.Venakides and X.Zhou, *Asymptotics for Polynomials Orthogonal with respect to Varying Exponential Weights*, Internat. Math. Res. Notices, no. **16**, 759-782, (1997) ; *Strong Asymptotics for Orthogonal Polynomials with respect to Varying Exponential Weights via Riemann-Hilbert Techniques*, to appear in Comm. Pure Appl. Math. ; *Uniform Asymptotics for Polynomials Orthogonal with respect to Varying Exponential Weights and Applications to Universality Questions in Random Matrix Theory*, to appear in Comm. Pure Appl. Math..

[DVZ1] P.A.Deift, S.Venakides and X.Zhou, *The collisionless shock region for the long-time behavior of solutions of the KdV equation*, Comm. Pure Appl. Math., **47** no. 2, 199–206, (1994).

[DVZ2] P.A.Deift, S.Venakides and X.Zhou, *New Results in Small Dispersion KdV by an Extension of the Steepest Descent Method for Riemann-Hilbert Problems*, Internat. Math. Res. Notices, no. **6**, 285-299, (1997).

[DZ1] P.A.Deift and X.Zhou, *A Steepest Descent Method for Oscillatory Riemman-Hilbert Problems; Asymptotics for the MKdV Equation*, Ann. Math., **137**, 295-368, (1993).

[DZ2] P.A.Deift and X.Zhou, *Asymptotics for the Painlevé II Equation*, Comm. Pure. Appl. Math., **48**, 277-337, (1995).

[FIK] A.S.Fokas, A.R.Its and V.E.Kitaev, *Discrete Painlevé equations and their appearance in quantum gravity*, Comm. Math. Phy., **142**, 313-344, (1991).

[Ge] I.M.Gessel, *Symmetric functions and P-recursiveness*, J. Combin. Theory. Ser. A, **53**, 257 - 285, (1990).

[Gr] C.Greene, *An extension of Schensted's theorem*, Adv. in Math., **14**, 254-265, (1974).

[IIKS] A.R.Its, A.G.Izergin, V.E.Korepin and N.A.Slavnov, *Differential equations for quantum correlation functions*, Int. J. Mod. Phy., **B4**, 1003-1037 ; *The quantum correlation functions as the τ function of classical differential equations*, 407-417, in *Important Development in Soliton Theory*, edited by A.S.Fokas and V.E.Zakharov, Springer-Verlag, 1993.

[Jo] K.Johansson, *The Longest Increasing Subsequence in a Random Permutation and a Unitary Random Matrix Model*, Math. Res. Lett., **5**, no. 1-2, 63-82, (1998).

[Ka] T.Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, 1976.

[Kn] D.E.Knuth, *The art of computer programming*, vol. 3 : *sorting and searching*, 2nd ed., Addison Wesley, Reading, Mass., 1973.

[Ra] E.M.Rains, *Increasing subsequences and the classical groups*, Electronic J. of Combinatorics, **5(1)**, R12, (1998).

[Sa] B.Sagan, *The Symmetric Group : Representations, Combinatorial Algorithms, and Symmetric Functions*, Wadsworth&Books/Cole, Pacific Grove, Calif., 1991.

[Sc] C.Schensted, *Longest increasing and decreasing subsequences*, Canad. J. Math., **13**, 179 - 191, (1961).

[Sz] G.Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, Vol 23, 4th Ed, New York, 1975.

[TW] C.A.Tracy and H.Widom, *Level-Spacing distributions and the Airy kernel*, Comm. Math. Phys., **159**, 151-174, (1994).